THE MATROID COMPLEX & ITS homology

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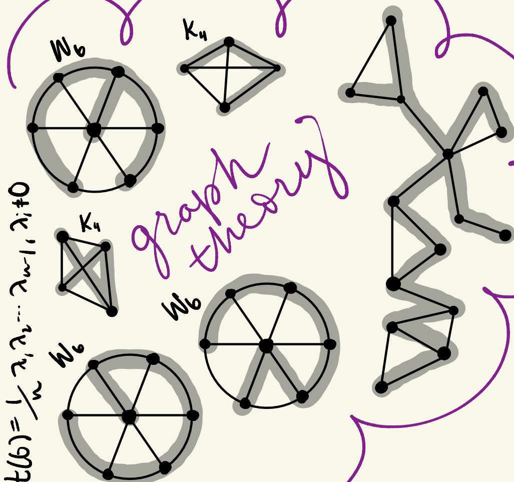
thanks to the MIDWEST RESEARCH EXPERIENCE FOR GRADUATES (MREG 2023)

Contents

Q1: What is a MATROID?

... and why do we care?



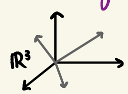
graph theory

$\tau(G) = \frac{1}{n} \lambda_1 \lambda_2 \cdots \lambda_{n-1}, \lambda_i \neq 0$

$\{e_1, e_2, \dots, e_n\} \subseteq \mathbb{R}^n$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in Mat_{\mathbb{Z}_3}(4,6)$$

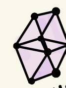
linear algebra



$\exists a_1, \dots, a_n \in \mathbb{C}^{\times}$ s.t. $\sum_{i=1}^{n} a_i v_i = 0$

Fix E/F
$S \subseteq E$ is algebraically independent
if $\nexists f(x_1, \dots, x_n) \in F[x_1, \dots, x_n] \setminus \{0\}$ s.t.
$0 = f(s_{i_1}, \dots, s_{i_n})\ \forall \{s_{i_1}, \dots, s_{i_n}\} \subseteq S$.

field theory



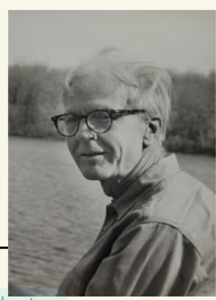
NON-ALGEBRAIC Vámos Matroid

ALGEBRAIC MATROIDS

$[\mathbb{Q}(t) : \mathbb{Q}] = \infty$
$S_1 = \{t\}, S_2 = \{t^2\}$ transcendence bases, but
$\mathbb{Q}(t^2) \subsetneq \mathbb{Q}(t)$

$[F(t) : F(P/Q)] = \max(\deg P, \deg Q)$

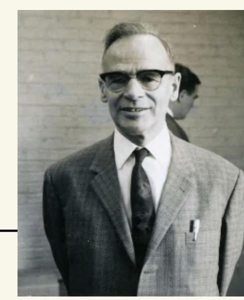
MATROIDS



HASSLER WHITNEY

1930's

B.L. van der WAERDEN



A.D.

+ Takeo Nakasawa, Saunders Mac Lane, W.T. Tutte, Richard Rado, & more!

definition 1.1 A **MATROID** is a pair $M=(E,\mathcal{B})$ for E a finite "ground set" and $\mathcal{B} \subseteq 2^E$ "bases" such that:

(B1) $\mathcal{B} \neq \emptyset$

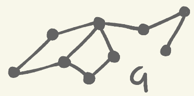
(B2) For $B_1, B_2 \in \mathcal{B}$, $B_1 \neq B_2$, and $x \in B_1 - B_2$, $\exists y \in B_2 - B_1$ then $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$ (Basis-Exchange Property)

Fact 1.1 All $B \in \mathcal{B}$ have the same cardinality

definition 1.2 We call $I \subseteq E$ **INDEPENDENT** if $\exists B \in \mathcal{B}$ such that $I \subseteq B$.

definition 1.3 The **RANK** $rk(M)$ of a matroid $M=(E,\mathcal{B})$ is given by $|B|$ for $B \in \mathcal{B}$. For any $X \subseteq E$, $rk(X) := \max\limits_{\substack{I \subseteq X \\ I \text{ independent}}} |I|$.

GRAPHIC MATROID



$\Longrightarrow$ $E = \{\text{edges of } G\}$

$\mathcal{B} = \{\text{spanning trees of } G\}$

LINEAR MATROID

$A := \left(a_1 \,|\, a_2 \,|\, \cdots \,|\, a_n \right) \in \mathrm{Mat}_{\mathbb{F}}(m,n)$ for some $\mathbb{F}$ $\Longrightarrow$ $E = \{1, 2, \dots, n\} = [n]$

$\mathcal{B} = \{ I \subseteq [n] : \{a_i\}_{i \in I} \text{ linearly independent over } \mathbb{F} \}$

↳ REGULAR MATROIDS are linear and realizable over <u>any</u> field $\mathbb{F}$.

ALGEBRAIC MATROID

Field Extension L/K $\Longrightarrow$ $E = \{e_1, \dots, e_k\} \subseteq L$

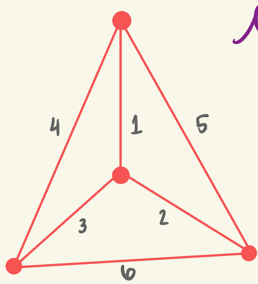
$\mathcal{B} = \{ I \subseteq E : \{e_i\}_{i \in I} \text{ algebraically independent over } K \}$

Irreducible Variety $V \subseteq K^n$ $\Longrightarrow$ $E = \{1, 2, \dots, n\}$

$\mathcal{B} = \{ S \subseteq E : \overline{\pi_S(V)} = K^{|S|} \}$

GRAPHIC $\subseteq$ REGULAR $\subseteq$ LINEAR $\subseteq$ ALGEBRAIC

example



W_3

edges

$$\mathcal{M}(W_3) = \big(\{1,2,3,4,5,6\},$$
$$\{\{1,2,3\}, \{1,2,4\}, \{1,2,6\}, \{1,3,5\}, \{1,3,6\}, \{2,3,4\},$$
$$\{2,3,5\}, \{1,4,5\}, \{2,5,6\}, \{3,4,6\}, \{2,4,5\}, \{2,4,6\},$$
$$\{1,4,6\}, \{1,5,6\}, \{3,4,5\}, \{3,5,6\}\} \big)$$

spanning trees

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

incidence matrix

other important features

definition 1.4 An element $x \in E$ is a **LOOP** if $x \notin B$ $\forall B \in \mathcal{B}$ (ex: $\vec{0} \in \mathbb{R}^3$)

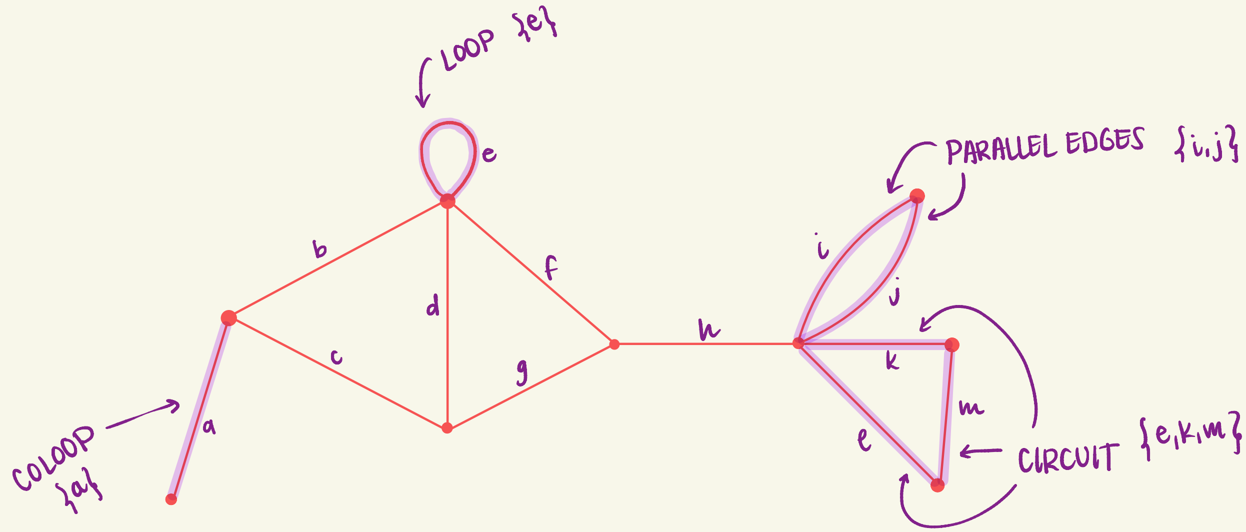
definition 1.5 An element $x \in E$ is a **COLOOP** if $x \in B$ $\forall B \in \mathcal{B}$

definition 1.6 A **CIRCUIT** is a minimally dependent set $C \subseteq E$. We denote the collection of circuits of a matroid $\mathcal{C}(M)$

definition 1.7 Two elements $x, y \in E$ are **PARALLEL** if $\{x, y\} \in \mathcal{C}(M)$.

Made with Goodnotes

example



A Graphic Matroid

definition 1.10 Two matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ are **ISOMORPHIC** if there exists a bijection $\varphi : E_1 \to E_2$ such that $B \in \mathcal{B}_1 \iff \varphi(B) \in \mathcal{B}_2$.

Q11: what is the MATROID COMPLEX?

... and why do we care?

Made with Goodnotes

recall

definition 2.1 A CHAIN COMPLEX (of vector spaces) consists of a family of $\mathbb{F}$-vector spaces $\{V_n\}_{n\in\mathbb{Z}}$ and a family of $\mathbb{F}$-linear maps $\partial = \partial_n : V_n \to V_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0 \Leftrightarrow \operatorname{im}\partial_n \subseteq \ker \partial_{n-1}$.

$$(V_\bullet, \partial_\bullet): \quad \cdots \longrightarrow V_{n+1} \xrightarrow{\partial} V_n \xrightarrow{\partial} V_{n-1} \xrightarrow{\partial} \cdots$$

When $\operatorname{im}\partial_n = \ker \partial_{n-1}$, we say $(V_\bullet, \partial_\bullet)$ is EXACT in degree n.

definition 2.2 The n^{th} HOMOLOGY $H_n(V_\bullet) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ measures the "non-exactness" in degree $n+1$.

Some history

1990s: Kontsevich introduces the GRAPH COMPLEX $(K_\bullet, \partial_\bullet)$

for $K_n := \mathbb{Q}\left\langle (\Gamma, \omega) : \begin{matrix}\Gamma \text{ connected admissible graph} \\ \omega \text{ edge orientation}\end{matrix} \right\rangle \Big/ \underset{\sigma : \Gamma \xrightarrow{\sim} \Gamma'}{(\Gamma, \omega) = \operatorname{sgn}(\sigma) \cdot (\Gamma', \omega)}$

∂ : edge contraction

$$\cdots \longrightarrow K_n \xrightarrow{\partial} K_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} K_4 \xrightarrow{\partial} K_3 \xrightarrow{\partial} K_2 \xrightarrow{\partial} K_1$$

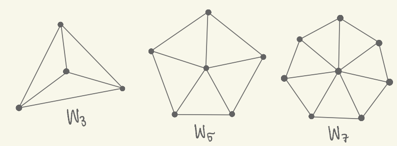
$\rightsquigarrow$ This is the "odd" version (related to invariants of even-dimensional manifolds)

$\rightsquigarrow$ Grothendieck Teichmüller Lie Algebra (DGLA)

more history

1990s : Kontsevich introduces the GRAPH COMPLEX $(K_\bullet, \partial_\bullet)$

2015 : Willwacher shows $[w_{2r+1}] \neq 0$ in graph homology



2020 : Chan, Galatius, Payne establish

$$\mathrm{Gr}^W_{6g-6} H^{4g-6-k}(\mathcal{M}_g;\mathbb{Q}) \cong \tilde{H}_{2g+k-1}(\Delta_g;\mathbb{Q}) \cong H_k(K^{(g)}_\bullet)$$

algebraic curves ⟷ tropical curves ⟷ graphs

Guiding Questions

$\emptyset$: Can matroid homology detect the graphic matroids associated to odd wheels?

$\emptyset$: Analogons correspondence in matroid setting?

top weight cohomology of $\mathcal{A}_g$?

Let (M, π) be the data of

- M a matroid on n ground set elements;
- π a total ordering of the ground set up to even permutation, i.e. $\pi = e_1 \wedge e_2 \wedge \ldots \wedge e_n \in (\bigwedge^n \mathbb{Z}^n)^\times$

The (rational) MATROID COMPLEX is given by $\{C_n\}_{n \in \mathbb{Z}}$ where

$$C_n = \mathbb{Q}\left\langle (M, \pi) \;\middle|\; \begin{array}{l} M \text{ a matroid} \\ |E(M)| = n \text{ and} \\ \pi \text{ an orientation} \end{array} \right\rangle \Big/ \begin{array}{c} (M, \pi) = \mathrm{sgn}(\sigma) \cdot (M', \pi) \\ \text{for } \sigma : M \xrightarrow{\sim} M' \end{array}$$

It follows that C_n admits the $\mathbb{Q}$-basis

$$\left\{ \begin{array}{c|l} \text{Isomorphism} & |E| = n, \ M \\ \text{classes } [M] & \text{admits no odd} \\ {\scriptstyle M=(E,\mathcal{B}) \text{ a matroid}} & \text{automorphism} \end{array} \right\}$$

Why "odd automorphisms"?

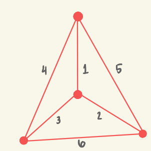
If M admits an odd automorphism, then

$$(M,\pi) = -(M,\pi) \Rightarrow [M] = 0 \in C_n$$

What are the differential maps ???

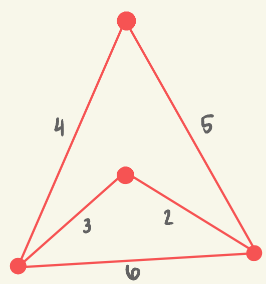
$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

example :



W_3

DELETION

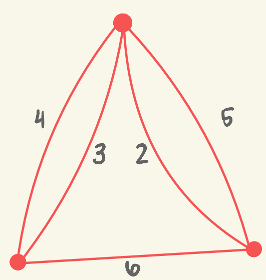


$W_3 \backslash 1$

$\Rightarrow$

$$M\backslash 1 = \Big(\{2,3,4,5,6\},$$
$$\Big\{ \cancel{\{1,2,3\}}, \cancel{\{1,2,4\}}, \cancel{\{1,2,6\}}, \cancel{\{1,3,5\}}, \cancel{\{1,3,6\}}, \{2,3,4\},$$
$$\{2,3,5\}, \cancel{\{1,4,5\}}, \{2,5,6\}, \{3,4,6\}, \{2,4,5\}, \{2,4,6\},$$
$$\cancel{\{1,4,6\}}, \cancel{\{1,5,6\}}, \{3,4,5\}, \{3,5,6\} \Big\} \Big)$$

CONTRACTION



$W_3 / 1$

$\Rightarrow$

$$M/1 = \Big(\{2,3,4,5,6\},$$
$$\Big\{ \{\cancel{1},2,3\}, \{\cancel{1},2,4\}, \{\cancel{1},2,6\}, \{\cancel{1},3,5\}, \{\cancel{1},3,6\}, \cancel{\{2,3,4\}},$$
$$\cancel{\{2,3,5\}}, \{\cancel{1},4,5\}, \cancel{\{2,5,6\}}, \cancel{\{3,4,6\}}, \cancel{\{2,4,5\}}, \cancel{\{2,4,6\}},$$
$$\{\cancel{1},4,6\}, \{\cancel{1},5,6\}, \cancel{\{3,4,5\}}, \cancel{\{3,5,6\}} \Big\} \Big)$$

differential maps

Let $M = (E, \mathcal{B})$.

definition 2.3 (Deletion) For $S \subseteq E$, the **DELETION** is the matroid given by $M \setminus S := (E \setminus S, \mathcal{B} \setminus S)$ where $\mathcal{B} \setminus S = \{ B \in \mathcal{B} \mid B \cap S = \emptyset \} = \mathcal{B} \cap 2^{E-S}$.

definition 2.4 (Duality) The **DUAL** matroid is given by $M^* := (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{ E - B \mid B \in \mathcal{B} \}$.

definition 2.6 (Contraction) For $S \subseteq E$, the **CONTRACTION** is the matroid given by $M/S := (M^* \setminus S)^*$.

We define the *deletion differential*

$$\partial_d : C_n \to C_{n-1}, \quad [M] \mapsto \sum_{\substack{k=1 \\ e_k \text{ noncoloop}}}^{n} (-1)^k [M \setminus e_k]$$

- $\partial_d^2 = 0$
- e_k must be noncoloop by (CB1)
- rank preserving

and obtain the matroid complex

$$(C_\bullet, \partial_d) : \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_d} C_n \xrightarrow{\partial_d} C_{n-1} \longrightarrow \cdots$$

And the rank-r subcomplex:

$$(C^r_\bullet, \partial_d) : \cdots \longrightarrow C^r_{n+1} \xrightarrow{\partial_d} C^r_n \xrightarrow{\partial_d} C^r_{n-1} \longrightarrow \cdots$$

For each n: $C_n \cong \bigoplus_{r=0}^{n} C_n^r$ AND $(C_\bullet, \partial_d) \cong \bigoplus_{r=0}^{n} (C_\bullet^r, \partial_d)$

By matroid duality:

$$\begin{tikzcd}
\cdots \arrow[r] & C_n^r \arrow[r, "\partial_d"] \arrow[d, "*"', "\wr"] & \cdots \arrow[r, "\partial_d"] & C_{r+2}^r \arrow[r, "\partial_d"] \arrow[d, "*"', "\wr"] \arrow[dr, phantom, "\circlearrowright"] & C_{r+1}^r \arrow[r, "\partial_d"] \arrow[d, "*"', "\wr"] & C_r^r \arrow[r, "\partial_d"] \arrow[d, "*"', "\wr"] & 0 \\
\cdots \arrow[r] & C_n^{n-r} \arrow[r, "\partial_c"] & \cdots \arrow[r, "\partial_c"] & C_{r+2}^2 \arrow[r, "\partial_c"] & C_{r+1}^1 \arrow[r, "\partial_c"] & C_r^0 \arrow[r, "\partial_c"] & 0
\end{tikzcd}$$

An isomorphism of chain complexes

Made with Goodnotes

Q III: What have we shown?

Proposition 1: Let M be a simple, rank 2 matroid on n ground set elements. Then $M \cong U_{2,n}$.

Pf: Each pair $\{x,y\} \subseteq E$ is independent, else $\{x,y\} \in \mathcal{C}(M)$, or x or y is a loop. $\square$

Theorem 2: $C^r_{r+1} \cong C^r_r \cong \begin{cases} \mathbb{Q} & r = 1 \\ 0 & r \neq 1 \end{cases}$

$C^r_{r+2} \cong 0 \quad \forall r \geq 0.$ (By Proposition 1)

computations

A description via Macaulay2's Matroids package:

$$\begin{tikzcd}
\cdots \arrow[r] & C_8 \arrow[r, "\partial d"] \arrow[d, phantom, "\cong" rotate=90] & C_7 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & C_6 \arrow[r, "\partial d"] \arrow[d, phantom, "\cong" rotate=90] & C_5 \arrow[r, "\partial d"] \arrow[d, phantom, "\cong" rotate=90] & \cdots \arrow[r, "\partial d"] & C_0 \arrow[d, phantom, "\cong" rotate=90] \\
\cdots \arrow[r] & \mathbb{Q}^{21} \oplus \mathbb{Q}^{299} \oplus \mathbb{Q}^{21} \arrow[r, "\partial d"] & \mathbb{Q}^{9} \oplus \mathbb{Q}^{9} \arrow[r, "\partial d"] & \mathbb{Q}^{2} \arrow[r, "\partial d"] & 0 \arrow[r, "\partial d"] & \cdots \arrow[r, "\partial d"] & 0
\end{tikzcd}$$

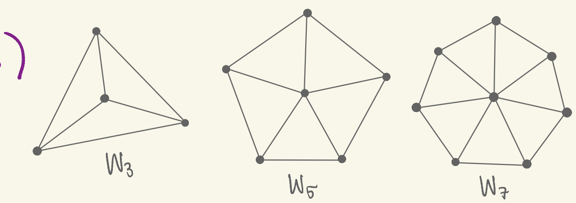
⚠ AND ⚠ $H_n(C_\bullet) = 0 \quad \forall n \leq 8$

(r,n)	0	1	2	3	4	5	6	7	8	9
0	0	$\mathbb{Q}$	0	0	0	0	0	0	0	0
1		$\mathbb{Q}$	$\mathbb{Q}$	0	0	0	0	0	0	0
2			0	0	0	0	0	0	0	0
3				0	0	0	$\mathbb{Q}^2$	$\mathbb{Q}^9$	$\mathbb{Q}^{21}$	$\mathbb{Q}^{131}$
4					0	0	0	$\mathbb{Q}^9$	$\mathbb{Q}^{299}$	$\mathbb{Q}^{174,771}$
5						0	0	0	$\mathbb{Q}^{21}$	$\mathbb{Q}^{174,771}$
6							0	0	0	$\mathbb{Q}^{131}$
7								0	0	0
8									0	0
9										0

Fig 1: C_n^r for $0 \leq r \leq n \leq 9$

results

Theorem 3: $[W_{2r-1}] = 0 \in H_{4r-2}(C_\bullet)$



pf: $[W_{2r-1} \cup \{\ell\}] \mapsto [W_{2r-1}] \mapsto 0.$

Theorem 4: (Acyclicity) $H_n(C_\bullet) = 0 \;\; \forall n$

pf: (strategy) Apply "loop-adjoining" technique from Thm 6.

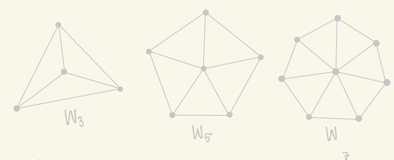
$\hookrightarrow$ dual to inflation on cones $\square$

$\boxed{\text{MORAL}}$ The whole matroid complex $C_\bullet$ is "too large", need to restrict to a "nice" subcomplex.

A good candidate is $R_\bullet$, the subcomplex of regular matroids.

more history

- 1990s: Kontsevich introduces the GRAPH COMPLEX $(K_\bullet, \partial_\bullet)$
- 2015: Willwacher shows $[w_{2r+1}] \neq 0$ in graph homology



- 2020: Chan, Galatius, Payne establish

$$Gr^W_{6g-6} H^{4g-6-k}(\mathcal{M}_g;\mathbb{Q}) \cong \tilde{H}_{2g+k-1}(\Delta_g;\mathbb{Q}) \cong H_k(K^{(g)}_\bullet)$$

- 2022: Brandt, Bruce, Chan, Melo, Moreland, Wolfe establish

$$Gr^W_{2d} H^{2d-i}(A_g;\mathbb{Q}) \cong \tilde{H}_{i-1}(LA_g^{trop,P};\mathbb{Q}) \cong H_{i-1}(P^{(g)}_\bullet) \text{ where } R^{(g)}_\bullet \subseteq P^{(g)}_\bullet$$

abelian varieties $\longleftrightarrow$ tropical varieties $\longleftrightarrow$ alternating perfect cones

Q IV: What would we still like to show?

Q: Can we compute homology of the subcomplex $R_\bullet$ generated by regular matroids?

Currently working on this in Oscar!

(r,n)	0	1	2	3	4	5	6	7	8	9
0	0	$\mathbb{Q}$	0	0	0	0	0	0	0	0
1		0	0	0	0	0	0	0	0	0
2			0	0	0	0	0	0	0	0
3				0	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0	0
4					0	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0
5						0	0	0	0	0
6							0	0	0	0
7								0	0	0
8									0	0
9										0

Fig 2: R_n^r for $0 \le r \le n \le 9$

Q: And understand how much larger $P_\bullet^{(g)}$ is compared to $R_\bullet^{(g)}$?

When $g = 2, 3$, $R_\bullet^{(g)} = P_\bullet^{(g)}$

Q: Can we equip $C_\bullet$ with a compatible graded Lie bracket $[-,-] : C_i \wedge C_j \to C_{i+j}$ to promote $C_\bullet$ to a differential graded Lie algebra?

thank you

"Tropical Curves, Graph Complexes, and Top-Weight Cohomology of $\mathcal{M}_g$"

Melody Chan, Søren Galatius, Sam Payne

"On the Top-Weight Rational Cohomology of $\mathcal{A}_g$"

Madeline Brandt, Juliette Bruce, Melody Chan, Margarida Melo, Gwyneth Moreland, Corey Wolfe